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MATHEMATICS

Accumulation of Errors in Approximate Calculations

by

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Presented by A. MOSTOWSKI on January 23, 1958

There is a growing need for a sufficiently general and easy method of computing the total error of rounding off for programs and subprograms for digital computers. In spite of a large number of papers on this matter [1]-[5] none of them seem to describe a method of this kind. The present note aims to sketch such a method used in the Mathematical Computers Laboratory in Warsaw. This method can be applied, e. g., in analogue computation.

Exact and approximate numerical sequences

The exact numerical sequence is defined as follows. We assume a sequence of initial values

$$x = \{x_k\} \qquad k = 1, 2, \ldots P \tag{1}$$

and on that basis we compute the sequence:

$$y = \{y_s\} \qquad s = 1, 2, \ldots N \tag{2}$$

in the following way:

$$1.\ \text{for}\quad s \leqslant P\colon\ y_s = x_s\,, \tag{3}$$

$$2.\ \text{for}\quad s > P\colon\ y_s = \omega_s(y_a, y_b, \ldots, y_d)\,, \tag{4}$$

where $a, b, \ldots, d < s$ and the functions ω_s have continuous partial derivatives in corresponding domains B_s.

The functions ω_s may represent *inter alia* arithmetical operations or the whole subprograms. One of the values y_s, e. g. y_N can be assumed as the aim of computation.

The functions ω_k being fixed, the y_N are functions of x, e. g. $y_N = y_N(x)$ with continuous partial derivatives with respect to every x_k.

If, for $P < s < N$, y_s is assumed to be independent variable, we have

$$y_N = y_{Ns}(x; y_s)\,. \tag{5}$$

The partial derivatives

$$\frac{\partial y_{Ns}(x; y_s)}{\partial y_s} \tag{6}$$

express the relation between y_N and the previously computed value y_s. It is worth mentioning that the knowledge of only $y_N(x)$ and $y_s(x)$ is not sufficient to determine y_{Ns}. To this aim it is necessary to know the proper ordering of operations involved in $y_N(x)$.

In common practice of computation the expression of initial values as well as computation of any functions involves an error, and as a result instead of the sequence (2) we have an approximate one. The values x_k in (1) are replaced by approximate values:

$$x_k^* = x_k + \delta x_k\,, \tag{7}$$

and the sequence $\{y_s^*\}$ is defined like $\{y_s\}$, namely:

(8) For $s \leqslant P$: $y_s^* = x_s^*$.

(9) For $s > P$: $y_s^* = \omega_s^*(y_a^*, \ldots, y_d^*) = \omega_s(y_a^*, \ldots, y_d^*) + \delta\omega_s$,

where $a, b, \ldots d$ have the same meaning as in (4), and $\delta\omega_s$ is an error related to computation of ω_s. The value

$$\delta y_N = y_N^* - y_N \tag{10}$$

is the total error yielded by computation of y_N.

The formula of superposition of errors

With the assumptions just stated above, we are going to prove that

$$\delta y_N = \sum_{k=1}^{P} \frac{\partial y_N(x)}{\partial x_k}\,\delta x_k + \sum_{s=P+1}^{N} \frac{\partial y_{Ns}(x; y_s)}{\partial y_s}\,\delta\omega_s + \sum_{k=1}^{P} a_k \delta x_k + \sum_{s=P+1}^{N} a_s \delta\omega_s\,, \tag{11}$$

where all the a_k and a_s tend to zero, whenever all the δx_k and $\delta\omega_s$ also tend to zero.

We notice at first that the value y_N considered as a function of x and all the errors made so far

$$y_N^* = y_N^*(x, \delta x, \delta\omega) \tag{12}$$

have continuous partial derivates with respect to all the variables. In fact, it follows from (7) and (8) that this is valid for $n \leqslant P$. If it is valid for $n \leqslant Q$, then it follows from (9) that, whenever $Q > P$, it is also valid for $n \leqslant Q+1$. Hence, our statement is valid for all n.

Hence, also, by the well-known theorem, the value

$$\delta y_N = y_N^*[x, \delta x, \delta\omega] - y_N^*[x, 0, 0], \tag{13}$$

which depends on δx, and $\delta\omega$ is of the form (11). It remains to be proved that the coefficients by δx and $\delta\omega$ in the first two members of the left side of (11) are the partial derivates as it is assumed there. It can be checked in the following manner. We assume that among δx_k and $\delta\omega_s$ only one value, e. g. $\delta\omega_s$ is different from zero. In this case (11) is reduced to two members, and the value of the coefficient by $\delta\omega_s$ can be easily verified.

If the functions ω_s have continuous partial derivates of second order, then the evaluation of a_k and a_s can be given. In most cases in common practice they are so small, that it can be assumed that

$$\delta y_N = \sum_{k=1}^{P} \frac{\partial y_N(x)}{\partial x_k} \delta x_k + \sum_{s=P+1}^{N} \frac{\partial y_{Ns}[x, y_s(x)]}{\partial y_s} \delta\omega_s . \tag{14}$$

This formula states simply that, putting apart increments of higher orders, all the errors accumulate linearly. It is worth mentioning that the values of derivates in this formula are quite independent from δx_k and $\delta\omega_s$.

Statistical analysis of accumulation of errors

From (14), e. g., the least upper bound of $|\delta y_N|$ can be easily evaluated, if the least upper bounds of $|\delta x_k|$ and $|\delta\omega_s|$ are known. However, the admission of this value as the total error would be, in case of a great number of constituent errors, too pessimistic. Therefore, the statistical estimation seems to be more adequate. To this aim we have to define the set of events, with respect to which the total error is a random variable. In the *analogue computation*, e. g. in case of sliding rule this set may consist of many computations made with the same scheme as initial values. In each of such computations, as the result of not exact measuring, errors δx_k and $\delta\omega_s$ are committed. Assuming that these errors are made at random with assumed distribution, we can analyze the probability distribution of δy_N. In numerical *digital computation* when the initial values, the way of computation, and the method of rounding off are fixed, the value of δy_N is determined uniquely. That is why we have to admit as a set of events the acts of computation with the same scheme, but slightly different initial values. Assuming that the initial values x' are taken at random from a small neighbourhood of x and the choice of any x' is equally probable, the errors δx_k and $\delta\omega_s$ can be considered as random variables also. For any x' from a sufficiently small neighbourhood of x the formula (14) can be applied. Replacing in this formula x' by x yields an error, which, by the assumption that all the derivates involved therein are continuous, is an increment of at

least second order with respect to $x'-x$, δx_k and $\delta\omega_s$. Thus, up to the increments of higher orders, the formula (14) with all derivates taken at the same point x can be admitted as a basis in our considerations.

Having one or another of these interpretations in mind, we assume that:

a) For purpose of analysis of the probability distribution of δy_N the formula (14) with fixed x is applied and the values δx_k and $\delta\omega_s$ are considered as random variables with probability distribution $\Phi_k(\delta x_s)$ and $\Psi_s(\delta\omega_s)$, respectively.

Hence,

$$M[\delta y_N] = \sum_{k=1}^{P} \frac{\partial y_N}{\partial x_k} \cdot M[\delta x_k] + \sum_{s=1}^{N} \frac{\partial y_{Ns}}{\partial y_s} \cdot M[\delta\omega_s],$$

where $M[\delta x_k]$ denotes the mean value of δy_N, etc.

Further we assume that:

b) All the δx_k and $\delta\omega_s$ are statistically independent. This assumption in analogue computations means that the values of successive errors are mutually independent. In digital computation this means that there is no systematic relation between successive values of error of rounding off. By the assumptions a and b we have

$$D[\delta y_N] = \sum_{k=1}^{P} \left[\frac{\partial y_N}{\partial x_k}\right]^2 \cdot D[\delta x_k] + \sum_{s=P+1}^{N} \left[\frac{\partial y_{Ns}}{\partial y_s}\right]^2 \cdot D[\delta\omega_s], \tag{15}$$

where $D[\delta x_k]$ denotes the variance of δx_k, etc.

Let us further assume that

c) There exists such an L such that for all k, s, N

$$\left|\frac{\partial y_N}{\partial x_k}\right| < L \quad \text{and} \quad \left|\frac{\partial y_{Ns}}{\partial y_s}\right| < L\,. \tag{16}$$

d) For all k and s such that $1 \leqslant k < P$, $s > P$

$$M[\delta x_k] = 0 \quad \text{and} \quad M[\delta\omega_s] = 0\,. \tag{17}$$

e) Each of the probability distributions is either normal or, for a fixed K, the condition

$$|\delta x_k| > K \quad \text{or} \quad |\delta\omega_s| > K \tag{18}$$

implies, respectively,

$$\Phi_k(\delta x_k) = 0 \quad \text{or} \quad \Psi_k(\delta\omega_s) = 0\,. \tag{19}$$

f) When $N \to \infty$,

$$D[\delta y_N] \to \infty\,. \tag{20}$$

Then, as $N \to \infty$, we have uniformly with respect to u

$$P\{|\delta y_N| < \sqrt{D[\delta y_N]} \cdot u\} \to \sqrt{\frac{2}{\pi}} \int_0^u \exp\left(-\frac{t^2}{2}\right) dt . \tag{21}$$

This formula follows from the well-known Lindeberg-Feller theorem. It can be also proved with somewhat weaker conditions concerning independence of δx_k and $\delta \omega_k$. The formula can be admitted as a basis for the practical analysis of the probability distribution of error.

EXAMPLE I. Let us assume that a polynomial

$$z = z(t) = a_0 t^n + a_1 t^{n-1} + \ldots + a_n$$

was computed according to Horner's scheme

$$\left.\begin{array}{l} t, a_0, \ldots a_n \\ \\ a_0 \cdot t + a_1 = y_1 \ . \ . \ . \\ . \ . \ . \ . \ . \ . \ . \ . \\ . \ . \ . \ . \ . \ . \ . \ . \\ y_{n-1} t + a_n = y_n = z(t) \end{array}\right\} \begin{array}{l} \text{initial values} \\ \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{computed values} \end{array}$$

On that basis we compute

$$\frac{\partial z}{\partial t} = z'(t) \qquad \frac{\partial z}{\partial a_k} = t^{n-k} \qquad \frac{\partial y_{Ns}}{\partial y_s} = \frac{\partial z}{\partial y_s} = t^{n-k} .$$

Let us further assume that the values t and a_k resulting from a foregoing computation step, have errors, which are considered as random variables with variances

$$D[\delta t] \quad \text{and} \quad D[\delta a_k] = D[\delta a] \quad (k = 0, 1, \ldots, n) .$$

Let us assume that the random errors yielded by computation of successive y_s have a constant variance $D[\delta \omega]$. Then, by assumption a and b and from (20), we obtain:

$$D[\delta z] = [z'(t)]^2 \cdot D[\delta t] + \frac{1 - t^{2n+2}}{1 - t^2} \{D[\delta a] + D[\delta \omega]\} .$$

$D[\delta z]$ being known and, assuming that conditions c) to f) are practically satisfied, we can apply (21) to evaluation of distribution of δz.

If in this example the values a_k are constant, i. e. they are coefficients of expanding of $\exp(t)$, then the assumption that a_k are random is not valid. However, the error of rounding off δa_k $(k = 0, \ldots, n)$, that is

$$(\delta a_0) t^n + (\delta a_1) t^{n-1} + \ldots + \delta a_n$$

can be computed by the aid of (14). The above polynomial may be evaluated and sketched on paper. It gives the systematic error of a given computation.

EXAMPLE II. Let the function $\varphi(t)$ be differentiable on $[0, 1]$ and

$$0 < \varphi(t) < 1 \qquad |\varphi'(t)| < m < 1 .$$

Then, as is well known, the following sequence

$$y_0 = x \qquad y_{k+1} = \varphi(y_k) \qquad 0 < x < 1$$

is convergent to a certain value y such that $y = \varphi(y)$. It can be easily checked, that

$$\left|\frac{\partial y_N}{\partial x}\right| < m^N \quad \text{and} \quad \left|\frac{\partial y_{Ns}}{\partial y_s}\right| < m^{N-s} .$$

Let x be taken at random according to the probability distribution with variance $D[\delta x]$. The variance of error involved in computing $\varphi(y)$ is assumed to equal $D[\delta\varphi]$. Then, we have

$$D[\delta y_N] < m^{2N} \cdot D[\delta x] + \frac{1}{1-m^2} D[\delta\varphi] .$$

The first member at the right side tends towards zero as $N \to \infty$. The second one does not depend on N. This means that $D[\delta y_N]$ is always bounded.

Final remarks

This method was applied in preparing programs for automatic digital computers. For the subprograms such as $\sin x$, $\ln x$, one step in the Runge-Kutta method etc. the least upper bound and variance of error transferred and generated by this subprograms were determined by means of the first and the second member of (14) and (15). Accordingly, in this way the errors for each part of the program, and finally for the whole program, were computed. However, care must be taken that all assumptions made here are fully satisfied.

The mechanical application of this method in all cases may yield unexpected results. For instance, in case of differential equations with solutions increasing at a very slow rate the assumption of independence of successive rounded values may fail to be satisfied. In not-stabil problems with very large derivates (6) the assumption of their continuity in corresponding domains, e.g. the condition c) may not be founded. In case of Example II. the condition f) is not fulfilled. This may obstruct the application of (21). Sometimes, also the initial values are of such a special kind that the statistical interpretation given here, involving random choice of initial values with uniform distribution does not fit in. However, in general, the analysis of error founded on the method presented above, is very useful.

The author is indebted to K. Bochenek and M. Fisz for helpful suggestions in preparing this paper.

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MATHEMATICS

On the Extension of Linear Transformations in Banach Spaces

by

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Presented by S. MAZUR on January 25, 1958

The present note contains some theorems concerning the extension of linear transformations in Banach spaces. It is known that the Hahn-Banach extension theorem for linear functionals in Banach spaces is, in general, not valid for linear transformations in such spaces. In this note we shall consider a class of linear transformations having this extension property. By application of these theorems we obtain also a known result concerning the extension of bilinear functionals in Banach spaces. A further application of the extension theorems for linear transformations enables us to treat the generalized moment problem for the case, when instead of a linear functional, a linear transformation is to be found which assumes the values given *a priori*. The method of proof is based on the Hahn-Banach extension theorem mentioned above; the notion of the tensor product of Banach spaces is not used.

1. Let X be a Banach space and $\bar{X}$ — its conjugate. Let $\mathfrak{K}$ be the class of all finitely dimensional linear *) operators from X into itself, i. e. the class of all operators of the form

$$Kx = \sum_{i=1}^{n} \varphi_i x \cdot x_i , \tag{1}$$

where $x_i \in X$, $\varphi_i \in \bar{X}$ $(i = 1, 2, \ldots, n)$. The norm of $K \in \mathfrak{K}$ is

$$\|K\| = \sup_{\|x\| \leq 1} \|Kx\| .$$

*) The word "linear" always means "additive and continuous".

The trace of an operator $K \in \mathfrak{K}$ of the form (1) is the number

$$\operatorname{tr}(K) = \sum_{i=1}^{n} \varphi_i x_i , \tag{2}$$

which does not depend on the representation of K in the form (1).

If A is a linear operator from X into itself, then the superposition AK belongs to $\mathfrak{K}$ and

$$\operatorname{tr}(AK) = \sum_{i=1}^{n} \varphi_i A x_i . \tag{3}$$

We shall say that A is a Fredholm operator if there exists a constant M such that

$$|\operatorname{tr}(AK)| \leqslant M\|K\| \tag{4}$$

for every $K \in \mathfrak{K}$.

Let now X be a subspace of the Banach space X_1. Every operator $K \in \mathfrak{K}$ of the form (1) can be extended to a finitely dimensional linear operator $\widetilde{K} \in \mathfrak{K}_1$ of the form

$$\widetilde{K}x = \sum_{i=1}^{n} \widetilde{\varphi}_i x \cdot x_i , \qquad x_i \in X , \tag{5}$$

where $\widetilde{\varphi}_i \in \bar{X}_1$ and $\widetilde{\varphi}_i$ is an extension of φ_i $(i = 1, 2, \ldots, n)$. (Such an extension is possible in virtue of the Hahn-Banach theorem). We have evidently

$$\|K\| \leqslant \|\widetilde{K}\| . \tag{6}$$

Putting $A\widetilde{K}x = \sum_{i=1}^{n} \widetilde{\varphi}_i x \cdot Ax_i$ we find that the superposition $A\widetilde{K}$ belongs to $\mathfrak{K}_1$ and

$$\operatorname{tr}(A\widetilde{K}) = \operatorname{tr}(AK) . \tag{7}$$

Thus, condition (4) implies, by (6) and (7), the inequality

$$|\operatorname{tr}(A\widetilde{K})| \leqslant M\|\widetilde{K}\| . \tag{8}$$

It is easy to see that condition (4) implies, too

$$\|A\| \leqslant M . \tag{9}$$

In this section we shall consider only Fredholm endomorphisms, i. e. linear endomorphisms satisfying condition (4). For such endomorphisms we have the following extension theorem.

THEOREM 1. *Every Fredholm endomorphism A of a reflexive Banach space X can be extended to an arbitrary reflexive Banach space X_1 containing X, i. e. if X_1 is an arbitrary reflexive Banach space containing X, then there exists a linear endomorphism $\widetilde{A}$ of X_1 such that $\widetilde{A}x = Ax$ for every x of X. Moreover, $\|\widetilde{A}\| \leqslant M$.*

Proof. It follows from conditions (7) and (8) that the additive functional $\mathrm{tr}(A\widetilde{K})$ defined on a linear subset of $\mathfrak{R}_1$ is bounded. Hence, by the Hahn-Banach theorem, it can be extended to a linear functional $F(K)$ defined for every K of $\mathfrak{R}_1$ with the norm

$$\|F\| \leqslant M\,. \tag{10}$$

Since the linear functional F is defined on the whole set $\mathfrak{R}_1$ of all finitely dimensional linear operators of X_1 of the form (5) with $x_i \in X_1$, it determines a bilinear functional $F(\varphi, x)$ defined on $\overline{X}_1 \times X_1$ as follows:

$$F(\varphi, x) = F(K_{\varphi,x}); \qquad x \in X_1; \qquad \varphi \in \overline{X}_1\,, \tag{11}$$

where the linear operator $K_{\varphi,x}$ is defined by the formula

$$K_{\varphi,x}(z) = \varphi z \cdot x, \qquad z \in X_1\,. \tag{12}$$

It follows from (10), (11) and (12) that

$$|F(\varphi, x)| \leqslant M\|\varphi\| \cdot \|x\|\,. \tag{13}$$

Thus, for fixed x of X_1, $F(\varphi, x)$ is a linear functional on $\overline{X}_1$. Since X_1 is reflexive, there exists an element $\widetilde{A}x$ of X_1 such that

$$F(\varphi, x) = \varphi(\widetilde{A}x) \quad \text{for every} \quad \varphi \text{ of } \overline{X}_1\,. \tag{14}$$

Formula (14) determines a linear transformation $\widetilde{A}$ of X_1 into itself. Condition (13) shows that $\|\widetilde{A}\| \leqslant M$. It remains to show that $\widetilde{A}$ is an extension of A. Since the linear functional $F(K)$, $K \in \mathfrak{R}_1$ is an extension of $\mathrm{tr}(AK)$, we have

$$F(K) = \mathrm{tr}(AK) \tag{15}$$

for every K of the form (5). It results from (15), (14), (12), (11) and (2) that

$$\varphi(\widetilde{A}x) = F(\varphi, x) = F(K_{\varphi,x}) = \mathrm{tr}(AK_{\varphi,x}) = \varphi(Ax)$$

for every φ of $\overline{X}_1$ and x of X. Hence we get

$\widetilde{A}x = Ax$ for every x of X and thus the theorem is proved.

Remark. Let A be a linear transformation of X with values in $X_1 \supset X$. Then A can be extended to X_1 if condition (4) is satisfied for every $K \in \mathfrak{R}_1$ of the form (1), where φ_i are linear functionals on $\overline{X}_1$ and x_i belong to X, provided that X_1 is reflexive. The norm of the extension does not exceed M.

Notice that the class of Fredholm endomorphisms coincides with the class of linear operators satisfying the Fredholm determinant theory developed by Leżański [1], since this theory can be applied to operator A, if and only if, the operator A satisfies the following condition ([1], p. 36):

There exists a linear functional F on the Banach space of linear endomorphisms of X such that

$$F(K) = \varphi_0 A x_0$$

for every operator of the form

$$Kx = \varphi_0 x \cdot x_0 \qquad (\varphi_0 \in \bar{X};\ x, x_0 \in X).$$

2. Let $\bar{X}_0$ be a subspace of the space $\bar{X}$ conjugated to a Banach space X. Let $\bar{\mathfrak{R}}$ be the class of all finitely dimensional linear operators K from $\bar{X}$ into itself of the form

$$K\varphi = \sum_{i=1}^{n} \varphi x_i \cdot \varphi_i, \tag{16}$$

where $x_i \in X$, $\varphi_i \in \bar{X}$ $(i = 1, 2, \ldots, n)$. The norm of $K \in \bar{\mathfrak{R}}$ is

$$\|K\| = \sup_{|\varphi| \leqslant 1} \left| \sum_{i=1}^{n} \varphi x_i \cdot \varphi_i \right|.$$

The trace of an operator $K \in \bar{\mathfrak{R}}$ of the form (16) is the number

$$\operatorname{tr}(K) = \sum_{i=1}^{n} \varphi_i x_i.$$

Denote by $\bar{\mathfrak{R}}_0$ the subclass of $\bar{\mathfrak{R}}$ of all operators of the form (16) provided that φ_i belong to $\bar{X}_0$.

Let B be a linear transformation from $\bar{X}_0$ into $\bar{X}$. The superposition BK belongs to $\bar{\mathfrak{R}}$ and

$$\operatorname{tr}(BK) = \sum_{i=1}^{n} B\varphi_i x_i.$$

Let us assume that the operator B satisfies the following condition:

There exists a constant M such that

$$|\operatorname{tr}(BK)| \leqslant M \|K\| \tag{17}$$

for every K of $\bar{\mathfrak{R}}_0$.

THEOREM 2. *If the linear operator B from $\bar{X}_0$ into $\bar{X}$ satisfies condition* (17), *then B can be extended to the whole $\bar{X}$ with norm not exceeding M, i. e. there exists a linear operator $\tilde{B}$ from $\bar{X}$ into itself such that $B\varphi = \tilde{B}\varphi$ for every φ of $\bar{X}_0$ and $\|\tilde{B}\| \leqslant M$.*

The proof of this theorem is analogous to that of Theorem 1.

3. Let X_0 be a subspace of a Banach space X; consider the class $\mathfrak{K}$ of all finitely dimensional linear operators K from X into $\bar{X}$ of the form

$$K\varphi = \sum_{i=1}^{n} \varphi(y_i)x_i\,, \tag{18}$$

where $x_i, y_i \in X$ and $\varphi_i \in \bar{X}$ $(i = 1, 2, \ldots, n)$. The norm of $K \in \mathfrak{K}$ is

$$\|K\| = \sup_{|\varphi|\leqslant 1} \Big\| \sum_{i=1}^{n} \varphi(y_i)x_i \Big\|.$$

Denote by $\mathfrak{K}_0$ the linear subset of $\mathfrak{K}$ of all operators of the form (18), provided that x_i, y_i belong to X_0.

Let C be a linear transformation from X_0 into $\bar{X}$. The superposition CK determines a finitely dimensional linear operator

$$CK\varphi = \sum_{i=1}^{n} \varphi(y_i)\, Cx_i, \qquad K \in \mathfrak{K}_0$$

from $\bar{X}$ into itself. The trace of operator CK is the number

$$\operatorname{tr}(CK) = \sum_{i=1}^{n} Cx_i(y_i)\,.$$

Let us assume that operator C satisfies the following condition:

There exists a constant M such that

$$|\operatorname{tr}(CK)| \leqslant M\|K\| \tag{19}$$

for every K of $\mathfrak{K}_0$.

THEOREM 3. *If the linear operator C from X_0 into $\bar{X}$ satisfies condition* (19), *then C can be extended to the whole space X with norm not exceeding M, i. e. there exists a linear operator $\widetilde{C}$ from X into $\bar{X}$ such that $\widetilde{C}x = Cx$ for every x of X_0 and $\|\widetilde{C}\| \leqslant M$.*

To prove this theorem we use the same argument as in the proof of Theorem 1.

Analogously we have

THEOREM 4. *If the linear operator C from X_0 into $\bar{X}_0$ satisfies condition* (19) *for every K being a linear operator from $\bar{X}_0$ into X_0 of the form* (18), *then C can be extended to the whole space X with norm not exceeding M, i. e. there exists a linear operator $\widetilde{C}$ from X into $\bar{X}_0$ such that $\widetilde{C}x = Cx$ for every x of X_0 and $\|\widetilde{C}\| \leqslant M$.*

By application of Theorem 3 we obtain the following theorem on extension of bilinear functionals proved by Marinescu [2]:

An arbitrary functional $\Phi(y, x)$ defined on a subset E of X can be extended to the whole space X if the following condition is satisfied:

There exists a constant M such that

$$(20)\qquad \Big|\sum_{l=1}^{p}\sum_{i=1}^{n}\sum_{k=1}^{m}\lambda_{il}\mu_{kl}\Phi(y_{il}, x_{kl})\Big| \leqslant M \sup_{f_1, f_2 \in \bar{X}} \frac{\sum_{l=1}^{p} f_1\big(\sum_{i=1}^{n}\lambda_{il}y_{il}\big)\cdot f_2\big(\sum_{k=1}^{m}\mu_{kl}x_{kl}\big)}{\|f_1\|\cdot\|f_2\|}.$$

Denote by $L(E)$ the linear manifold spanned by the elements of E and put

$$x_l = \sum_{k=1}^{m}\mu_{kl}x_{kl}, \qquad y_l = \sum_{i=1}^{n}\lambda_{il}y_{il}.$$

It follows from condition (20) that $\Phi(y_l, x_l)$ for $x_l, y_l \in L(E)$ is a bilinear functional on $L(E)$, where $\Phi(y_l, x_l)$ is equal to the left-hand side of (20). This bilinear functional determines a linear operator C from $L(E)$ into its conjugate $L(E)^*$, where $C(x_l)$ is the linear functional $\Phi(y_l, x_l)$ defined for $y_l \in L(E)$ and fixed x_1.

We have

$$\sum_{l=1}^{p}\Phi(y_l, x_l) = \sum_{l=1}^{p} Cx_l(y_l) = \operatorname{tr}\Big(\sum_{l=1}^{p} f_1(y_l)\,Cx_l\Big) = \operatorname{tr}(CK),$$

where $K \in \mathfrak{R}_0$ is a linear operator from $\bar{X}$ into $L(E)$ defined by the equation

$$Kf_1 = \sum_{l=1}^{p} f_1(y_l)x_l, \qquad f_1 \in \bar{X}.$$

Since the right-hand side of condition (20) is equal to the norm of $K\,(=\|K\| = \sup_{\|f_1\|\leqslant 1}\|Kf_1\|)$, conditions (19) and (20) are equivalent. Hence, by Theorem 4, there exists an extension $\widetilde{C}$ of the linear operator C to the whole space X. The linear operator $\widetilde{C}$ from X into $L(E)^*$ determines a bilinear functional on $L(E)\times X$ which is an extension of Φ and can be extended in the same way on the basis of Theorem 3.

Notice that the proof of this theorem given by Marinescu is based on the notions of tensor product and cross-spaces investigated by Schatten [3].

4. Let X be a reflexive Banach space and S an abstract set. The problem consists in finding a linear operator A from X into itself such that $Ax(s) = y(s)$, where $x(s)$ and $y(s)$, $s \in S$, are elements of X given *a priori*.

THEOREM 5. *Let us assume that the following condition is satisfied: There exists a constant M such that*

$$\left|\sum_{i=1}^{n}\sum_{k=1}^{m}\mu_{ik}\varphi_i y(s_{ik})\right| \leqslant M \sup_{\|x\|\leqslant 1}\left\|\sum_{i=1}^{n}\sum_{k=1}^{m}\mu_{ik}\varphi_i(x)x(s_{ik})\right\|, \tag{21}$$

where the numbers μ_{ik}, n, m and $s_{ik} \in S$ $\varphi_i \in \bar{X}$ are arbitrarily chosen.

Then there exists a linear operator A from X into itself such that $Ax(s) = y(s)$ for every s of S and $\|A\| \leqslant M$.

Proof. Condition (21) permits us to determine a linear operator A on the linear subspace spanned by the elements $x(s)$, $s \in S$ with values $y(s) = Ax(s)$ for every s of S.

Condition (21) means

$$\left|\sum_{i=1}^{n}\varphi_i Ax_i\right| \leqslant M\|K\|,$$

where $x_i = \sum_{k=1}^{m}\mu_{ik}x(s_{ik})$ and the operator K is defined by the formula

$$Kx = \sum_{i=1}^{n}\varphi_i(x)x_i.$$

Hence, by Theorem 1 operator A can be extended to the whole space X with norm not exceeding M.

Let X be an arbitrary Banach space and $\bar{X}$ — its conjugate. Let $\varphi(s)$ and $\psi(s)$, $s \in S$ be linear functionals on X indexed by an abstract set S. The following theorem gives a sufficient condition for the existence of a linear operator B from $\bar{X}$ into itself such that $B\varphi(s) = \psi(s)$ for every s of S.

THEOREM 6. *Let us assume that the following condition is satisfied: There exists a constant M such that*

$$\left|\sum_{i=1}^{n}\sum_{k=1}^{m}\mu_{ik}\psi(s_{ik})x_i\right| \leqslant M \sup_{\|\varphi\|\leqslant 1}\left\|\sum_{i=1}^{n}\sum_{k=1}^{m}\mu_{ik}\varphi x_i \cdot \varphi(s_{ik})\right\|,$$

where the numbers μ_{ik}, n, m and $s_{ik} \in S$, $x_i \in X$ are arbitrarily chosen.

Then there exists a linear operator B from $\bar{X}$ into itself such that $B\varphi(s) = \psi(s)$ for every s of S and $\|B\| \leqslant M$.

The proof of this theorem is based on Theorem 2.

Let $x(s) \in X$ and $\psi(s) \in X$ for $s \in S$ be elements indexed by an abstract set S. Applying Theorem 3, we obtain a sufficient condition for the existence of a linear operator C from X into $\bar{X}$ such that $Cx(s) = \psi(s)$ for every s of S. We have, namely

THEOREM 7. *Let us assume that the following condition is satisfied: There exists a constant M such that*

$$\left|\sum_{i=1}^{n}\sum_{k=1}^{m}\mu_{ik}\psi(s_{ik})y_i\right| \leqslant M\sup_{|\varphi|\leqslant 1}\left\|\sum_{i=1}^{n}\sum_{k=1}^{m}\mu_{ik}\varphi(y_i)\cdot x(s_{ik})\right\|,$$

where the numbers μ_{ik}, n, m and $s_{ik}\in S$, $y_i\in X$ are arbitrarily chosen.

Then there exists a linear operator C from X into $\bar{X}$ such that $Cx(s)=\psi(s)$ for every s of S and $\|C\|\leqslant M$.

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MATHEMATICS

Some Remarks on Conjugate Spaces Containing Subspaces Isomorphic to the Space c_0

by

C. BESSAGA and A. PEŁCZYŃSKI

Presented by S. MAZUR on February 25, 1958

We shall use the standard terminology introduced by Banach in [1]. We shall denote by m — the space of all real bounded sequences $x=(a_n)$ with the norm $|x| = \sup_n |a_n|$, by c_0 — the subspace of m consisting of all sequences convergent to 0, by l — the space of all sequences $f = (a_n)$ such that $\sum_{n=1}^{\infty} |a_n| < \infty$ with the norm $\|f\| = \sum_{n=1}^{\infty} |a_n|$. We shall denote by X^* and X^{**} the first and the second conjugate space of the space X respectively.

THEOREM 1. *The following conditions are equivalent*:

(i) *the space X^* contains a subspace isomorphic to c_0,*

(ii) *the space X^* contains a subspace isomorphic to m,*

(iii) *the space X contains a complementary subspace *)isomorphic to l,*

(iiii) *there exists a linear mapping of the space X onto l.*

According to Theorem 1 of [4] we obtain

COROLLARY 1. *If the space X contains no complementary subspace isomorphic to l, especially if every bounded set in X is conditionally weakly compact* **) [3], then in X^* every weakly unconditionally convergent series is unconditionally convergent* ***).

As $X \subset X^{**}$, we obtain

COROLLARY 2. *If the space X contains a subspace isomorphic to c_0 then the space X contains a complementary subspace isomorphic to l.*

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*) The subspace Y of the space X is said to be a complementary subspace if there exists a projection operation X onto Y.

**) i. e. if every sequence of elements of this set contains a weakly convergent subsequence (not necessarily to an element of X).

***) For the definition of unconditional and weakly unconditional convergence of series see e. g. [4].

COROLLARY 3. *In order that for every separable B-space E there exist a linear mapping of the space X onto E, it is necessary and sufficient that X contains a complementary subspace isomorphic to l.*

Proof. The necessity follows from the equivalence of conditions (iii) and (iiii). The sufficiency follows from the fact that every separable B-space is a linear image of l ([2], p. 111).

COROLLARY 4. *The conjugate space X^* contains no complementary subspace isomorphic to c_0.*

The idea of the proof. This corollary follows from the proof of Theorem 1. We show that if X^* contains the space E isomorphic to c_0, then there exist subspaces E_0 and E_1 such that E_0 is isomorphic to c_0 and is complemented in E, E_1 is isomorphic to m and $E_0 \subset E_1 \subset X^*$. Now, applying the result of Philips ([5], 7.5) we obtain our corollary.

COROLLARY 5. *If the space X^* contains a subspace isomorphic to c_0 then the spaces X^* and $X^* \times c_0$ *) are not isomorphic but their first conjugate spaces are isomorphic.*

This corollary follows immediately from Theorem 1, Corollary 4 and the fact that the space $l \times l$ is isomorphic to l.

Remark. The first conjugate space of the spaces m and $m \times c_0$ are isometric if we admit in the space $m \times c_0$ the new norm,

$$|||(x_1, x_2)||| = \max(\|x_1\|, \|x_2\|) .$$

The proof of Theorem 1 will be published elsewhere [6].

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*) By $X \times Y$ we denote the Cartesian product of the spaces X and Y.

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MATHEMATICS

A Connection between Weakly Unconditional Convergence and Weakly Completeness of Banach Spaces

by

A. PEŁCZYŃSKI

Presented by S. MAZUR on February 25, 1958

W. Orlicz has proved in [2] that if the B-space X is weakly complete *) then every weakly unconditionally convergent series is unconditionally convergent. In the first part of our note we consider the class of B-spaces in which the converse implication holds. For example every subspace of the space with an absolute basis belongs to this class. In the second part we define an operation of u^*-closure. Using this notion we obtain some criterion of the weakly completeness of conjugate spaces.

The space with the property (u)

Definition 1. A B-space X will be said to have the *property* (u) if for every weakly convergent sequence $(x_n) \subset X$ there is a sequence $(y_n) \subset X$ such that:

a) the series $\sum_{n=1}^{\infty} y_n$ is weakly unconditionally convergent,

b) the sequence $x_n - \sum_{i=1}^{n} y_i$ weakly converges to 0.

Proposition 1. *If in a space X having the property (u) weakly unconditional convergence is equivalent to unconditional convergence then X is weakly complete.*

Proposition 2. *Every weakly complete space has the property (u). In particular every reflexive space has the property (u).*

Theorem 1. *If the space X has the property (u) then X is weakly complete if, and only if, no subspace of X is isomorphic to c_0.*

Theorem 2. *The space X has the property (u) if, and only if, every separable subspace of X has the property (u).*

*) The terminology and notations used in this paper are the same as in the paper "Some Remarks on Conjugate Spaces..." (this issue).

COROLLARY 1. *Every subspace of the space with the property* (u) *has the property* (u).

THEOREM 3. *If the space* X *has an absolute basis then* X *has the property* (u).

COROLLARY 2. *Every subspace of the space with an absolute basis, has the property* (u).

PROPOSITION 3. *There exists the space* J *which has not the property* (u); *there is the semi-reflexive space given by R. C. James in* [1].

COROLLARY 3. *No space which contains a subspace isomorphic to* J *has the property* (u). *In particular the space* C *of all real continuous functions on the interval* $\langle 0; 1\rangle$ *has not the property* (u).

COROLLARY 4. *The space* C *can be imbedded in no B-space with an absolute basis.*

The operation of u^*-closure

Let Z be a linear subset of the conjugate space Y^*.

Definition 2. The functional $f \in Y^*$ is an u_1^*-*limit-point* of the set Z if for every separable subspace $E \subset Y$ there exists a sequence $(f_n) \subset Z$ such that for every $x \in E$ the following conditions are satisfied.

$$\sum_{n=1}^{\infty} |f_n(x)| < \infty . \tag{1}$$

$$\sum_{n=1}^{\infty} f_n(x) = f(x) . \tag{2}$$

The set of all u_1^*-limit-points of Z will be denoted by $u_1^*(Z)$. Suppose that for every ordinal number $\beta < \alpha < \Omega$ we have defined the set $u_\beta^*(Z)$; we define the set $u_\alpha^*(Z)$ as follows

$$u_\alpha^*(Z) = \sum_{\beta<\alpha} u_\beta^*(Z) .$$

Let us put

$$u^*(Z) = \sum_{\alpha<\Omega} u_\alpha^*(Z) .$$

In the sequel we shall deal with the case, where $Z = X$, $Y = X^*$ and $Y^* = X^{**}$.

THEOREM 4. *If* $u^*(X) = X^{**}$ *then the space* X^* *is weakly complete.*

PROPOSITION 4. *If the space* X *has the property* (u) *and every bounded set in* X *is conditionally weakly compact then* $u_1^*(X) = u^*(X) = X^{**}$.

COROLLARY 5. *If the space* X *has the property* (u) *and every bounded set in* X *is conditionally weakly compact then* X^* *is weakly complete.*

Remark. For all known B-spaces if X^* is weakly complete then $u^*(X) = X^{**}$. In the case of the space $C(Q)$ of all continuous real functions on the compact Q we have $u^*(X) = u_2^*(X) = X^{**}$.

The proofs will appear in Studia Mathematica.

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MATHÉMATIQUE

Sur les valeurs extrémales du module de la dérivée des fonctions univalentes bornées

par

W. JANOWSKI

Présenté par K. KURATOWSKI le 25 février 1958

Considérons les fonctions holomorphes et univalentes dans le cercle $|z|<1$ de la forme

$$F(z)=z+A_2z^2+\dots \tag{1}$$

Soit $M>1$ un nombre positif quelconque, F_M — la famille de toutes les fonctions bornées de la forme (1) assujetties à la condition $|F(z)|<M$, et F_∞ — la famille de toutes les fonctions de la forme (1).

La valeur de $z\neq 0$ étant fixée, nous considérons la fonctionnelle

$$|F'(z)| \tag{2}$$

définie dans la famille F_∞ et par conséquent dans F_M.

Je démontre les théorèmes suivants:

THÉORÈME 1. *Pour les fonctions de la famille F_M l'estimation suivante, dont les limites sont atteintes, a lieu:*

$$A(|z|, M)\leqslant |F'(z)|\leqslant B(|z|, M)$$

où, pour chaque point $(|z|, M)$ du domaine

$$D_0\{0<|z|<1,\ M>1\},$$

$$A(|z|, M)=\frac{x}{|z|}\,\frac{1+\dfrac{x}{M}}{1-\dfrac{x}{M}}\,\frac{1-|z|}{1+|z|},$$

$x\left(0<x<(\sqrt{2}-1)M\right)$ est la solution de l'équation

$$\frac{\left(\dfrac{x}{M}+1\right)^2}{x}=\frac{(|z|+1)^2}{|z|}.$$

L'expression $B(|z|, M)$ *est définie comme il suit:*

1° *pour chaque point* $(|z|, M)$ *du domaine*

$$D_1\{0 < |z| < 1,\ M > M_1(|z|)\}, \quad \text{où} \quad M_1(|z|) = \max\left(1, \frac{|z|}{(1-|z|)^2}\cdot\frac{[1-h(|z|)]^2}{h(|z|)}\right),$$

$$h(|z|) = -1 + \frac{1+|z|}{\sqrt{2|z|}},$$

$$B(|z|, M) = \frac{y}{|z|}\,\frac{1-\frac{y}{M}}{1+\frac{y}{M}}\,\frac{1+|z|}{1-|z|},$$

$y(0 < y < M\cdot h(|z|)$ *est la racine de l'équation*

$$\frac{\left(\frac{y}{M}-1\right)^2}{y} = \frac{(|z|-1)^2}{|z|}$$

2° *pour chaque point* $(|z|, M)$ *de l'ensemble*

$$D_2\{\tfrac{1}{2} < |z| < 1,\ M_2(|z|) < M \leqslant M_1(|z|)\}, \quad \text{où} \quad M_2(|z|) = (2|z|)^3,$$

$$\log B(|z|, M) = \log 2Mu\,\frac{1-\left(\frac{u}{M}\right)^2}{1+\left(\frac{u}{M}\right)^2} + \frac{\sqrt{\left(\frac{u}{M}\right)^2+2\left(\frac{u}{M}\right)-1}}{1+\frac{u}{M}}\log\frac{1}{|z|}\,\frac{1+\frac{u}{M}-\sqrt{\left(\frac{u}{M}\right)^2+2\left(\frac{u}{M}\right)-1}}{1+\frac{u}{M}+\sqrt{\left(\frac{u}{M}\right)^2+2\left(\frac{u}{M}\right)-1}},$$

$u\left(M\cdot h(|z|) \leqslant u < \frac{1}{2}M\right)$ *est la solution de l'équation*

$$\log\frac{\left(\frac{u}{M}-1\right)^2}{2u\left[\left(\frac{u}{M}\right)^2+2\left(\frac{u}{M}\right)-1\right]} + \frac{1+\frac{u}{M}}{\sqrt{\left(\frac{u}{M}\right)^2+2\left(\frac{u}{M}\right)-1}}\times$$

$$\times\log|z|\,\frac{1+\frac{u}{M}+\sqrt{\left(\frac{u}{M}\right)^2+2\left(\frac{u}{M}\right)-1}}{1+\frac{u}{M}-\sqrt{\left(\frac{u}{M}\right)^2+2\left(\frac{u}{M}\right)-1}} = 0.$$

3° *pour chaque point* $(|z|, M)$ *de l'ensemble*

$$D_3\{\tfrac{1}{2} < |z| < 1,\ 1 < M \leqslant M_2(|z|)\}$$

$$\log B(|z|, M) = \log M \frac{1-\left(\frac{v}{M}\right)^2}{1-|z|^2} + \frac{v^2}{M^2-v^2} \log \frac{v}{M \cdot |z|}$$

$v(\frac{1}{2} M \leqslant v \leqslant M)$ *est la racine de l'équation*

$$\log M + \frac{M^2 - v^2}{v^2} \log \frac{v}{M \cdot |z|} = 0 .$$

THÉORÈME 2. *Pour les fonctions de la famille* F_∞ *l'éstimation suivante, dont les limites sont atteintes, a lieu:*

$$A(|z|, \infty) \leqslant |F'(z)| \leqslant B(|z|, \infty)$$

où

$$A(|z|, \infty) = \frac{x}{|z|} \frac{1-|z|}{1+|z|}, \qquad B(|z|, \infty) = \frac{y}{|z|} \frac{1+|z|}{1-|z|}$$

et

$$x = \frac{|z|}{(1+|z|)^2}, \qquad y = \frac{|z|}{(1-|z|)^2} .$$

Les résultats obtenus pour les fonctions de la famille F_∞ coïncident avec ceux de L. Bieberbach, obtenus par d'autres procédés [1].

Avant d'aborder la démonstration il y a lieu de remarquer que les théorèmes conservent leur généralité si l'on suppose, que $z = r > 0$. Cette remarque est utilisée dans la démonstration.

Remarquons en outre, que les théorèmes cités sont équivalents à ceux qui établissent l'existence des fonctions extrémales par rapport à la fonctionnelle (2) dans la famille F_M resp. dans F_∞. Le module de chacune de ces fonctions est égal dans un certain point à $A(|z|, M)$ où $B(|z|, M)$ resp. $A(|z|, \infty)$ et $B(|z|, \infty)$. Pour démontrer ces derniers théorèmes considérons la famille F_T de toutes les fonctions univalentes dans le cercle $|z| < 1$ de la forme

$$f(z) = a_1 z + a_2 z^2 + \dots ,$$

où $a_1 \geqslant T$, T étant un nombre arbitraire de l'intervalle (0,1). Supposons que ces fonctions sont assujetties à la condition $|f(z)| < 1$ pour tout $|z| < 1$. Si la valeur $0 < r < 1$ est fixée, on peut considérer pour chaque $f(z)$ de la famille F_T la fonctionnelle

$$k(f) = \left[\frac{f'(z)}{a_1}\right]_{z=r}, \tag{3}$$

qui y est determinée.

On voit immédiatement, que la recherche des limites précises du module $|F'(r)|$ des fonctions $F(z) \in F_M$ est tout à fait équivalente au problème analogue relatif aux expressions (3) dans la famille F_T, si l'on pose $T = 1/M$.

Je constate d'abord que l'on peut appliquer à la fonctionnelle (3) les résultats de Z. Charzyński et W. Janowski [2] relatifs aux fonctions extrémales par rapport à la fonctionnelle (3) dans la famille F_T. Je démontre ensuite, que chaque fonction $w = f^*(z)$ de la famille F_T, qui est extrémale par rapport à la fonctionnelle (3) satisfait à l'équation

$$(4)\qquad \frac{w'^2}{w^2}\left[\frac{2w_3w-w_3^2}{(w-w_3)^2}+\frac{2w_4w-w^2}{(w-w_4)^2}+\lambda\right]= \\ =\frac{1}{z^2}\left[\frac{2z_3z-z_3^2}{(z-z_3)^2}+\frac{2z_4z-z^2}{(z-z_4)^2}+k\frac{z(z_3-z_4)}{(z-z_3)(z-z_4)}+\lambda\right]$$

où

$$w_3=f^*(r),\quad w_4=\frac{1}{\overline{w}_3},\quad z_3=r,\quad z_4=\frac{1}{r},\quad k=r\frac{f^{*\prime\prime}(r)}{f^{*\prime}(r)},\quad \lambda=-\frac{p^*\cdot T}{|f^{*\prime}(r)|}$$

et

$$p^*=\frac{f^{*\prime}(r)}{T}\min_{0\leqslant y\leqslant 2\pi}\left[\frac{2w_3e^{iy}-w_3^2}{(e^{iy}-w_3)^2}+\frac{2w_4e^{iy}-e^{2iy}}{(e^{iy}-w_4)^2}\right].$$

En profitant des propriétés de l'équation générale des fonctions extrémales, on peut présenter l'équation (4) sous l'une des formes suivantes: pour $\lambda \neq 1$

$$(I)\qquad \frac{w'^2}{w^2}\frac{(w-w_0)^2(w-w_1)(w-w_2)}{(w-w_3)^2(w-w_4)^2}=\frac{1}{z^2}\frac{(z-z_0)^2(z-z_1)(z-z_2)}{(z-z_3)^2(z-z_4)^2}$$

où

$$|w_0|=1,\quad |w_1|<1,\quad |z_0|=1,\quad |z_1|<1,\quad w_2=\frac{1}{\overline{w}_1},\quad z_2=\frac{1}{\overline{z}_1};$$

$$(II)\qquad \frac{w'^2}{w^2}\frac{(w-w_0)^2(w-w_1)(w-w_2)}{(w-w_3)^2(w-w_4)^2}=\frac{1}{z^2}\frac{(z-z_0)^2(z-z_1)^2}{(z-z_3)^2(z-z_4)^2}$$

où

$$|w_0|=1,\quad |w_1|<1,\quad |z_0|=1,\quad |z_1|=1,\quad w_2=\frac{1}{\overline{w}_1};$$

$$(III)\qquad \frac{w'^2}{w^2}\frac{(w-w_0)^2(w-w_1)^2}{(w-w_3)^2(w-w_4)^2}=\frac{1}{z^2}\frac{(z-z_0)^2(z-z_1)^2}{(z-z_3)^2(z-z_4)^2}$$

où

$$|w_0|=1,\quad |w_1|=1,\quad |z_0|=1,\quad |z_1|=1;$$

pour $\lambda = 1$,

$$\text{(IV)} \qquad \frac{w'^2}{w^2}\,\frac{2w_3(w-w_0)^2 w}{(w-w_3)^2(w-w_4)^2} = \frac{1}{z^2}\,\frac{a(z-z_0)^2 z}{(z-z_3)^2(z-z_4)^2}$$

où

$$|w_0| = 1, \qquad |z_0| = 1, \qquad a = 2z_3 + k(z_3 - z_4)\,.$$

En vertu de la condition $w(0) = 0$ et vu la définition du nombre w_3 on peut déterminer les nombres w_0, w_1, z_0 et z_1 à l'aide de $|w_3|$ et de r. On obtient trois systèmes des valeurs w_0, w_1, z_0, z_1 dans le cas de l'équation (I) et un système dans les cas (II)-(IV).

L'intégration des équations (I)-(IV) pour chaque système de w_0, w_1, z_0, z_1 et l'étude des équations obtenues dans l'entourage des points $z = 0$ et $z = r$ conduisent à la détermination du module $|w_3|$ en fonction de r et T et du module $|f^{*\prime}(r)|$ en fonction de r, T et $|w_3|$ dans chaque point (r, T) du carré $D_0\{0 < r < 1, 0 < T < 1\}$.

On obtient pour chaque point $(r, T) \in D_0$ deux valeurs extrémales de la fonctionnelle $|f'(r)|$, dont l'une présente le minimum, l'autre le maximum de cette fonctionnelle.

Enfin, en normalisant les fonctions de la famille F_T par leur premier coefficient, nous obtiendrons aisément le théorème 1 et ensuite, en passant à la limite pour $M \to \infty$, le théorème 2.

Une démonstration détaillée des résultats précédents sera publiée dans le Bulletin de la Société des Sciences et des Lettres de Łódź, Classe III.

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The Spinor Space

by

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1. Introduction

We wish to investigate some of the consequences of the assumption that the physical space is an eight-dimensional spinor space rather than a four-dimensional vector space. The reasons for an assumption of this kind are the following:

1) The physical laws are governed by the Lorentz group and in this group spinors play a fundamental role rather than vectors.

2) The spinor space is richer in transformations not violating the principle of relativity than the vector space. One group of such transformations is the group of gauge transformations. Charge conservation gets thus its geometrical explanation and is put, therefore, on the same footing with energy-momentum and angular momentum. Another group of such transformations is the unitary group u_2 which transforms the spinor space without changing the Minkowski space and which may, possibly, be connected with isotopic spin transformations.

3) Besides, there are other transformations of the spinor which do not change the co-ordinates of the vector space and which may be useful for the description of the internal structure of the particles.

4) The spinor space is in a natural way contained in the mathematical framework of the Lorentz group. It seems, therefore, reasonable to investigate first all the possibilities provided by this group before one starts to look for other groups (vector spaces with more than four dimensions) in the mathematical description of sub-micro phenomena.

2. The connection between the spinor and the vector space

The most general connection between vectors (X_1, X_2, X_3, X_0) and second rank spinors $z_{\alpha\dot{\beta}}$ $(\alpha = 1, 2;\ \beta = 1, 2)$ is well known to be

$$
\begin{aligned}
z_{1\dot{1}} &= X_3 - X_0 & X_1 &= \tfrac{1}{2}(z_{1\dot{2}} + z_{2\dot{1}}) \\
z_{2\dot{2}} &= -X_3 - X_0 & X_2 &= \frac{i}{2}(z_{1\dot{2}} - z_{2\dot{1}}) \\
z_{1\dot{2}} &= X_1 - iX_2 & X_3 &= \tfrac{1}{2}(z_{1\dot{1}} - z_{2\dot{2}}) \\
z_{2\dot{1}} &= X_1 + iX_2 & X_0 &= -\tfrac{1}{2}(z_{1\dot{1}} + z_{2\dot{2}})\,.
\end{aligned}
\tag{2.1}
$$

We get, therefore, the most general connection between vectors and first rank spinors $z_\alpha, z_{\dot{\alpha}}$ $(\alpha = 1, 2)$ if we assume that the $z_{\alpha\dot{\beta}}$ are some functions of z_α and $z_{\dot{\alpha}}$ and their complex conjugates.

$$
z_{\alpha\dot{\beta}} = z_{\alpha\dot{\beta}}(z_1, z_2, z_{\dot{1}}, z_{\dot{2}}, z_1^*, z_2^*, z_{\dot{1}}^*, z_{\dot{2}}^*)\,. \tag{2.2}
$$

Assuming that the physical space is the eight-dimensional space of the four complex (eight real) variables $z_\alpha, z_{\dot{\alpha}}$ we have in (2.2) a connection of this space with the conventional space of the variables X_μ.

In (2.2) there occur four unknown functions $z_{\alpha\dot{\beta}}$ determining this connection. If we do not wish to introduce from the beginning undetermined functions into the theory, we have to decide upon the functional dependence of $z_{\alpha\dot{\beta}}$ on $z_\alpha, z_{\dot{\alpha}}$. We shall make here the simplest assumption that the $z_{\alpha\dot{\beta}}$ are bilinear forms of the $z_\alpha, z_{\dot{\beta}}$

$$
z_{\alpha\dot{\beta}} = a z_\alpha z_{\dot{\beta}} + b z_{\dot{\alpha}}^* z_\beta^* + c z_\alpha z_\beta^* + d z_{\dot{\alpha}}^* z_{\dot{\beta}} \tag{2.3}
$$

and we shall investigate the consequences of this assumption. Other, more involved relations between X_μ and $z_\alpha, z_{\dot{\alpha}}$ are, of course, *a priori*, also possible. However, by the choice of (2.3) we satisfy the "principle of simplicity".

The coefficients in (2.3) are complex numbers which must satisfy certain relations in order that $z_{1\dot{1}}$ and $z_{2\dot{2}}$ are real and that $z_{1\dot{2}}$ is complex conjugate to $z_{2\dot{1}}$. These relations are

$$
a = b^*, \qquad c = c^*, \qquad d = d^*\,. \tag{2.4}
$$

The form (2.3) may be further simplified (without loss of generality) by changing the units in which the co-ordinates z_α and $z_{\dot{\alpha}}$ are to be measured.

$$
z_\alpha \to r z_\alpha, \qquad z_{\dot{\alpha}} \to s z_{\dot{\alpha}}\,. \tag{2.5}
$$

The word "measured" is used here in an abstract meaning. The co-ordinates $z_\alpha, z_{\dot{\alpha}}$ are, in contradistinction to X_μ, certainly not observable quantities in a similar sense as the spinor functions ψ_α in the Dirac equation.

The change of units (2.5) together with condition (2.4) brings (2.3) into the form

$$rsaz_\alpha z_{\dot\beta} + r^* s^* a^* z^*_{\dot\alpha} z^*_\beta + c|r|^2 z_\alpha z^*_\beta + d|s|^2 z^*_{\dot\alpha} z_{\dot\beta} . \tag{2.6}$$

We choose now r and s in such a way that

$$rsa = r^* s^* a^* = \lambda, \qquad c|r|^2 = \epsilon\, d|s|^2 = \varkappa , \tag{2.7}$$

where λ and $\varkappa$ are two real parameters. The sign factor

$$\epsilon = \pm 1 \tag{2.8}$$

in the second of Eqs. (2.7) is necessary to account for the relative sign of c and d. Thus,

$$z_{\alpha\dot\beta} = \lambda(z_\alpha z_{\dot\beta} + z^*_{\dot\alpha} z^*_\beta) + \varkappa(z_\alpha z^*_\beta + \epsilon z^*_{\dot\alpha} z_{\dot\beta}) . \tag{2.9}$$

Apart from the possible changes of the parameters λ and $\varkappa$, Eq. (2.9) represents two forms: one for $\epsilon = +1$ and the other for $\epsilon = -1$. We shall decide upon one of these alternatives when considering the scalar product of two vectors.

3. Invariants

We shall now study the connection between the invariants in both spaces. For this purpose we introduce another vector U_μ connected with the second order spinor $u_{\alpha\dot\beta}$ in the same way as X_μ with $z_{\alpha\dot\beta}$ (Eqs. (2.1)), the $u_{\alpha\dot\beta}$ being bilinear combinations of u_α and $u_{\dot\alpha}$ of the type (2.9).

$$u_{\alpha\dot\beta} = \lambda'(u_\alpha u_{\dot\beta} + u^*_{\dot\alpha} u^*_\beta) + \varkappa'(u_\alpha u^*_\beta + \epsilon' u^*_{\dot\alpha} u_{\dot\beta}) . \tag{3.1}$$

With the help of (2.1) we get first

$$X_\mu U_\mu = -\tfrac{1}{2} z_{\alpha\dot\beta} u^{\alpha\dot\beta}, \tag{3.2}$$

where components with upper indices are obtained from those with lower indices according to the rules

$$z^1 = z_2, \qquad z^2 = -z_1 \tag{3.3}$$

and similarly for the dotted indices. Einstein's convention about summation over dummy indices is adopted as well for the vector indices μ (from 1 to 4, $X_4 = iX_0$) as for the spinor indices α, β (from 1 to 2).

Introducing now (3.1) and (2.9) into (3.2) we get

$$X_\mu U_\mu = -\operatorname{Re}\Big\{\lambda\lambda'(ab + cd) + \frac{\varkappa\varkappa'}{2}(|a|^2 + \epsilon\epsilon'|b|^2 + \epsilon'|c|^2 + \epsilon|d|^2) + \\ + \lambda\varkappa'(ad^* + \epsilon' bc^*) + \lambda'\varkappa(ac + \epsilon bd)\Big\}, \tag{3.4}$$

where a, b, c, d are the invariants (with respect to the unimodular group c_2)

$$a = z_\alpha u^\alpha, \quad b = z_{\dot\alpha} u^{\dot\alpha}, \quad c = z^*_\alpha u^{\dot\alpha}, \quad d = z^*_{\dot\alpha} u^\alpha. \tag{3.5}$$

In the particular case when

$$z_\alpha = u_\alpha, \quad z_{\dot\alpha} = u_{\dot\alpha} \tag{3.6}$$

and

$$\lambda = \lambda', \quad \varkappa = \varkappa', \quad \epsilon = \epsilon', \tag{3.7}$$

we get

$$X_\mu = U_\mu, \quad a = b = 0, \quad c = -d^* = z^*_\alpha z^{\dot\alpha}, \tag{3.8}$$

and relation (3.4) takes the simple form

$$X^2_\mu = (\lambda^2 - \epsilon\varkappa^2)\,|z^*_\alpha z^{\dot\alpha}|^2. \tag{3.9}$$

From (3.9) it is seen immediately that for $\epsilon = -1$ we have only the possibility to represent space-like points by means of spinor variables. Therefore we decide for the alternative

$$\epsilon = +1. \tag{3.10}$$

Thus we get for the connection of vectors and spinors the final formula

$$z_{\alpha\dot\beta} = \lambda(z_\alpha z_{\dot\beta} + z^*_{\dot\alpha} z^*_\beta) + \varkappa(z_\alpha z^*_\beta + z^*_{\dot\alpha} z_{\dot\beta}), \tag{3.11}$$

and for the connection between the invariants

$$X^2_\mu = (\lambda^2 - \varkappa^2)\,|z^*_\alpha z^{\dot\alpha}|^2. \tag{3.12}$$

Let us now consider the formulae (3.11-12) more closely.

1) For fixed $|z^*_\alpha z^{\dot\alpha}|^2$ we get all the hyperboloids $-\infty < X^2_\mu < +\infty$ by changing continuously the parameters $\varkappa$ and λ. In particular for $\lambda^2 > \varkappa^2$ we get all the hyperboloids in the outside of the light cone and for $\lambda^2 < \varkappa^2$ all hyperboloids in the inside of the light cone. For $\lambda^2 = \varkappa^2$ we get the light cone $X^2_\mu = 0$. We may obtain all points on one particular hyperboloid by a rotation in the X_μ-space or, which is the same, by a transformation of the group c_2 in the spinor space z_α, $z_{\dot\alpha}$. Thus we may arrive at all points of the Minkowski space by a transformation which leaves $z^*_\alpha z^{\dot\alpha}$ invariant (c_2-group) and a change of the parameters $\varkappa$ and λ.

2) There is also another way to arrive at all points of the Minkowski manifold for fixed λ and $\varkappa$. Of course these parameters must be chosen differently inside and outside the light cone. Outside the light cone we must have $\lambda^2 > \varkappa^2$, e. g.

$$\lambda^2 = 1, \quad \varkappa^2 = 0. \tag{3.13}$$

Inside the light cone we must demand $\lambda^2 < \varkappa^2$, e. g.

$$\lambda^2 = 0, \quad \varkappa^2 = 1. \tag{3.14}$$

Assuming for definitness the choice (3.13-14) we have for the corresponding relation between the invariants

$$X_\mu^2 = \pm |z_\alpha^* z^{\dot\alpha}|^2, \tag{3.15}$$

where the upper (lower) sign corresponds to points in the outside (inside) of the light cone. We arrive now at all points of one hyperboloid by a transformation of the c_2-group and we go over from one hyperboloid to another arbitrary hyperboloid of the same side of the light cone by changing continuously $|z_\alpha^* z^{\dot\alpha}|^2$ form 0 to ∞. This may always be achieved by a proper change of the variables $z_\alpha, z_{\dot\alpha}$. These changes are, of course, not transformations of c_2 since they change the value of the invariant $z_\alpha^* z^{\dot\alpha}$.

In particular we get the points of the light cone by putting either $z_\alpha = 0$ $(z_{\dot\alpha} \neq 0)$ or $z_{\dot\alpha} = 0$ $(z_\alpha \neq 0)$ or $z_{\dot\alpha} = \varrho z_\alpha^*$ with ϱ an arbitrary complex number. These three examples are of importance, since they show the necessity of an eight-dimensional spinor space if one wishes to represent all points of Minkowski's manifold. Indeed the only way to reduce the number of variables in the spinor space is to introduce some relations between these variables. The only relations which are consistent with the transformation properties of spinors (c_2-group) are just the three mentioned above and they lead to a geometry on the light cone only. Such a geometry may (but need not) be used for the description of massless particles. The need for a four-dimensional spinor space (two-component spinors) in the description of events at the light cone arises when we assume explicitly that $\varkappa^2 \neq \lambda^2$ that is, e. g. in the case (3.15).

4. Inversions of space and time

Inversions $I_{\vec r}$ of space and I_t of time induce the following transformations of the second order spinors

$$I_{\vec r}: \quad X_i' = -X_i, \ (i = 1, 2, 3), \ X_0' = X_0; \qquad \begin{matrix} z'_{1\dot 1} = z_{2\dot 2} & z'_{1\dot 2} = -z_{1\dot 2} \\ z'_{2\dot 1} = -z_{2\dot 1} & z'_{2\dot 2} = z_{1\dot 1} \end{matrix} \tag{4.1}$$

$$I_t: \quad X_i' = X_i, \ (i = 1, 2, 3), \ X_0' = -X_0; \qquad \begin{matrix} z'_{1\dot 1} = -z_{2\dot 2} & z'_{1\dot 2} = z_{1\dot 2} \\ z'_{2\dot 1} = z_{2\dot 1} & z'_{2\dot 2} = -z_{1\dot 1} \end{matrix} \tag{4.2}$$

In view of the general relation (3.11) these inversions may be realized by a linear transformation of the eight variables $z_\alpha, z_{\dot\alpha}, z_\alpha^*, z_{\dot\alpha}^*$ with properly chosen coefficients. We shall not go into a detailed discussion of the transformation matrix but shall restrict ourselves to some particular

examples. Thus it is easily verified, that, e. g. the transformations

$$(4.3)\qquad \begin{matrix} z_1' = az_2 & z_1' = a^{-1}z_2 \\ z_2' = -az_1 & z_2' = -a^{-1}z_1 \end{matrix} \quad \text{or} \quad \begin{matrix} z_1' = az_2^* & z_1' = a^{-1}z_2^* \\ z_2' = -az_1^* & z_2' = -a^{-1}z_1^* \end{matrix}$$

are inversions of space I_r^- if a satisfies the condition

$$(4.4)\qquad |a|^2 = 1\,.$$

It may be shown further that no transformations of the variables $z_a, z_a^{\cdot}, z_a^*, z_a^{\cdot *}$ can be found which would represent inversions of time unless we change simultaneously the value of the parameter $\varkappa$ from $\varkappa$ to $-\varkappa$. Thus it is readily verified that, e. g. a simultaneous transformation

$$(4.5)\qquad \begin{matrix} z_1' = az_2^{\cdot} & z_1' = -a^{-1}z_2 \\ z_2' = -az_1^{\cdot} & z_2' = a^{-1}z_1 \end{matrix} \quad \text{or} \quad \begin{matrix} z_1' = az_2^* & z_1' = -a^{-1}z_2^{\cdot *} \\ z_2' = -az_1^* & z_2' = a^{-1}z_1^{\cdot *} \end{matrix}$$

and

$$(4.6)\qquad \varkappa' = -\varkappa$$

represents an inversion of time, if condition (4.4) is satisfied. This lets us suspect that the sign of the real parameter $\varkappa$ differentiates between the vectors of the future and past light cones. We shall verify this suspicion in the next Section. It may be noted that the choice $\epsilon = -1$ is responsible for the above situation since it gives the two forms $z_1 z_1^* + \epsilon z_1^{\cdot *} z_1^{\cdot}$ and $z_2 z_2^* + \epsilon z_2^{\cdot *} z_2^{\cdot}$ a positive definite character. The choice $\epsilon = -1$ would not help, however. Indeed, apart from the fact that for $\epsilon = -1$ we have to do with space-like points only, inversions of space would now demand a change of the relative sign of $\varkappa$ and λ.

5. Movements of the spinor space which leave the vector space unchanged

Let us write explicitly the formulae (2.1) and (3.11) connecting the two spaces.

$$(5.1)\qquad \begin{aligned} X_1 &= \frac{\lambda}{2}(z_1 z_2^{\cdot} + z_1^{\cdot *} z_2^* + z_2 z_1^{\cdot} + z_2^* z_1^{\cdot *}) + \frac{\varkappa}{2}(z_1 z_2^* + z_1^{\cdot *} z_2^{\cdot} + z_2 z_1^* + z_2^{\cdot *} z_1^{\cdot}) \\ X_2 &= i\frac{\lambda}{2}(z_1 z_2^{\cdot} + z_1^{\cdot *} z_2^* - z_2 z_1^{\cdot} - z_2^* z_1^{\cdot *}) + i\frac{\varkappa}{2}(z_1 z_2^* + z_1^{\cdot *} z_2^{\cdot} - z_2 z_1^* - z_2^{\cdot *} z_1^{\cdot}) \\ X_3 &= \frac{\lambda}{2}(z_1 z_1^{\cdot} + z_1^{\cdot *} z_1^* - z_2 z_2^{\cdot} - z_2^* z_2^{\cdot *}) + \frac{\varkappa}{2}(z_1 z_1^* + z_1^{\cdot *} z_1^{\cdot} - z_2 z_2^* - z_2^{\cdot *} z_2^{\cdot}) \\ -X_0 &= \frac{\lambda}{2}(z_1 z_1^{\cdot} + z_1^{\cdot *} z_1^* + z_2 z_2^{\cdot} + z_2^{\cdot *} z_2^*) + \frac{\varkappa}{2}(z_1 z_1^* + z_1^{\cdot *} z_1^{\cdot} + z_2 z_2^* + z_2^{\cdot *} z_2^{\cdot})\,. \end{aligned}$$

For the following discussion it will be convenient to express these relations by means of the real variables $x_a, y_a, x_a^{\cdot}, y_a^{\cdot}$, $(a = 1, 2)$

$$(5.2)\qquad z_a = x_a + iy_a, \qquad z_a^{\cdot} = x_a^{\cdot} - iy_a^{\cdot}\,.$$

Introducing (5.2) into (5.1) we get easily

$$
\begin{aligned}
X_1 &= \lambda(x_1 x_{\dot 2} + y_1 y_{\dot 2} + x_{\dot 1} x_2 + y_{\dot 1} y_2) + \varkappa(x_1 x_2 + y_1 y_2 + x_{\dot 1} x_{\dot 2} + y_{\dot 1} y_{\dot 2}) \\
X_2 &= \lambda(x_1 y_{\dot 2} - y_1 x_{\dot 2} + x_{\dot 1} y_2 - y_{\dot 1} x_2) + \varkappa(x_1 y_2 - y_1 x_2 + x_{\dot 1} y_{\dot 2} - y_{\dot 1} x_{\dot 2}) \\
X_3 &= \lambda(x_1 x_{\dot 1} + y_1 y_{\dot 1} - x_2 x_{\dot 2} - y_2 y_{\dot 2}) + \frac{\varkappa}{2}(x_1^2 + y_1^2 + x_{\dot 1}^2 + y_{\dot 1}^2 - x_2^2 - y_2^2 - x_{\dot 2}^2 - y_{\dot 2}^2) \\
-X_0 &= \lambda(x_1 x_{\dot 1} + y_1 y_{\dot 1} + x_2 x_{\dot 2} + y_2 y_{\dot 2}) + \frac{\varkappa}{2}(x_1^2 + y_1^2 + x_{\dot 1}^2 + y_{\dot 1}^2 + x_2^2 + y_2^2 + x_{\dot 2}^2 + y_{\dot 2}^2).
\end{aligned} \tag{5.3}
$$

We may now carry out a simultaneous rotation among the real variables $x_\alpha, y_\alpha, x_{\dot\alpha}, y_{\dot\alpha}$ which brings the bilinear expressions representing X_3 and X_0 to the canonical form. This is a rotation about the angle $\pi/4$ given by the following formulae

$$
\begin{aligned}
x_\alpha &= \frac{1}{\sqrt{2}}(\xi_\alpha - \xi_{\dot\alpha}) & y_\alpha &= \frac{1}{\sqrt{2}}(\eta_\alpha - \eta_{\dot\alpha}) \\
x_{\dot\alpha} &= \frac{1}{\sqrt{2}}(\xi_\alpha + \xi_{\dot\alpha}) & y_{\dot\alpha} &= \frac{1}{\sqrt{2}}(\eta_\alpha + \eta_{\dot\alpha}).
\end{aligned} \tag{5.4}
$$

Introducing the new variables (5.4) into (5.3) we obtain finally

$$
\begin{aligned}
X_1 &= (\varkappa+\lambda)(\xi_1\xi_2 + \eta_1\eta_2) &&+ (\varkappa-\lambda)(\xi_{\dot 1}\xi_{\dot 2} + \eta_{\dot 1}\eta_{\dot 2}) \\
X_2 &= (\varkappa+\lambda)(\xi_1\eta_2 - \eta_1\xi_2) &&+ (\varkappa-\lambda)(\xi_{\dot 1}\eta_{\dot 2} - \eta_{\dot 1}\xi_{\dot 2}) \\
X_3 &= (\varkappa+\lambda)\tfrac{1}{2}(\xi_1^2 + \eta_1^2 - \xi_1^2 - \eta_1^2) + (\varkappa-\lambda)\tfrac{1}{2}(\xi_1^2 + \eta_1^2 - \xi_2^2 - \eta_2^2) \\
-X_0 &= (\varkappa+\lambda)\tfrac{1}{2}(\xi_1^3 + \eta_1^2 + \xi_2^2 + \eta_2^3) + (\varkappa-\lambda)\tfrac{1}{2}(\xi_1^2 + \eta_1^2 + \xi_2^2 + \eta_2^2).
\end{aligned} \tag{5.5}
$$

It is easily seen from (5.5) that every vector appears here as a sum of two zero-vectors. Indeed, the terms containing $\varkappa+\lambda$ form separately a vector on the light cone and similarly the terms containing $\varkappa-\lambda$. We notice further that for vectors of the inside of the light cone $\varkappa^2 > \lambda^2$ and we have simultaneously $\varkappa+\lambda > 0$ and $\varkappa-\lambda > 0$ for $\varkappa > \lambda$. Similarly, for $\varkappa < -\lambda$ we have simultaneously $\varkappa+\lambda < 0$ and $\varkappa-\lambda < 0$. In the first case the form representing $-X_0$ is positively definite in the second case it is negatively definite. Consequently, the range of values $\varkappa > \lambda$ corresponds to the past light cone and the range of values $\varkappa < -\lambda$ to the future light cone. This verifies the suspicion of Section 4.

So far we have emphasized those transformations of the variables $z_\alpha, z_{\dot\alpha}$ of the spinor space and possibly those transformations of the parameters $\lambda, \varkappa$ which correspond to some transformations of the co-ordinates X_μ of the vector space which are not the identity transformation. In fact we have shown that all movements in the Minkowski space may be obtained by properly chosen movements in the spinor space.

There arises the question whether there are some movements of the spinor space which have no counterpart in the vector space, i. e. move-

ments which leave the co-ordinates X_μ unchanged. Such movements would be of particular interest, giving us new possibilities already in frames of the conventional geometry with $X_\mu^2 =$ invariant.

If we take the X_μ to be constant, we have in (5.5) four equations for the eight real variables $\xi_\alpha, \xi_{\dot\alpha}^*, \eta_\alpha, \eta_{\dot\alpha}^*$ $(\alpha = 1, 2)$. These equations represent four seven-dimensional hypersurfaces of second order and their cut is in general a four-dimensional manifold. Any transformation which transforms one point of this manifold into another arbitrary point of the same manifold does not change, of course, the value of the $X_\mu - s$. It may be noted that the equations of this four-dimensional manifold and the corresponding transformations inside the manifold vary in general if we change the point X_μ. We shall give a detailed discussion of these manifolds and of general transformations which do not change the $X_\mu - s$ at another place. Here we want only to point out the simplest transformations of this type.

One such transformation is given by the formulae

$$\begin{aligned} \xi'_\alpha &= \xi_\alpha \cos\varphi - \eta_\alpha \sin\varphi \qquad \xi'^*_{\dot\alpha} = \xi^*_{\dot\alpha} \cos\chi - \eta^*_{\dot\alpha} \sin\chi \\ \eta'_\alpha &= \xi_\alpha \sin\varphi + \eta_\alpha \cos\varphi \qquad \eta'^*_{\dot\alpha} = \xi^*_{\dot\alpha} \sin\chi + \eta^*_{\dot\alpha} \cos\chi \end{aligned} \tag{5.6}$$

with two arbitrary angles φ and χ.

In terms of the complex variables $\zeta_\alpha = \xi_\alpha - i\eta_\alpha$ and $\zeta^*_{\dot\alpha} = \xi^*_{\dot\alpha} - i\eta^*_{\dot\alpha}$ (which transform with respect to transformations of the group c_2 in the same way as the variables $z_\alpha, z^*_{\dot\alpha}$) these transformations take the from

$$\zeta'_\alpha = e^{i\varphi}\zeta_\alpha, \qquad \zeta'^*_{\dot\alpha} = e^{i\chi}\zeta^*_{\dot\alpha}. \tag{5.7}$$

Taking $\chi = -\varphi$ we get the gauge transformations, taking $\chi = \varphi$ we get another commutative group of transformations which also does not change the variables X_μ.

For $\chi = -\varphi$ each of these two $(\alpha = 1, 2)$ unitary transformations may be interpreted as a rotation about an angle 2φ around some axis $\vec{r}_2$ in some three-dimensional vector space (not to be confused with the X_μ-space).

It is seen further that for time-like vectors, if we put $\lambda = 0$, there exists another simple group of transformations which do not change the $X_\mu - s$. This is the transformation

$$\begin{aligned} z'_\alpha &= e^{i\vartheta} z_\alpha \\ iz'^*_{\dot\alpha} &= e^{-i\vartheta} iz^*_{\dot\alpha} \end{aligned} \tag{5.8}$$

(cf. (3.14) and (5.1)). Each of the two transformations (5.8) may be interpreted as a rotation about an angle 2ϑ in the same three-dimensional vector space which corresponds to (5.7). The invariant axis $\vec{r}_3$ is ortho-

gonal to $\vec{r}_2$ ($\vec{r}_2\vec{r}_3 = 0$). It is well known that each rotation in the space $\vec{r}_2, \vec{r}_3, \vec{r}_1 = [\vec{r}_2 \cdot \vec{r}_3]$ may be obtained by three successive transformations of the type (5.7-8).

It is seen that the consideration of the simplest transformations of the spinor space which do not change the co-ordinates X_μ of the Minkowski space leads necessarily to the notion of three-dimensional rotations and gauge transformations. This fact provides a geometrical interpretation of charge and isotopic spin.

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EXPERIMENTAL PHYSICS

Polarization of Photoluminescence of Organophosphors

by

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The dependence of the degree of polarization of fluorescence on the wave length of exciting light was investigated by various authors [1]-[5]. A variation of the degree of polarization as a function of the exciting wave length was obtained, and, moreover, negative degrees of polarization were also observed. The change is brought about by transitions from the normal state to various electronic levels of the molecule in the act of light absorption, the virtual oscillators corresponding to these transitions being, in general, different from that pertaining to the emission band [3], [6]. In most cases, investigations of the above dependence were conducted with liquid and rigid solutions of dyes, yielding negligible intensities of phosphorescence light. However, if the quotient $\frac{\eta_P}{\eta_F}$ be not negligibly small within a given region of excitation (η_P denoting the yield of phosphorescence, and η_F — that of fluorescence), the observed polarization is no longer that of fluorescence alone, but rather that of a mixture of the fluorescence light and the phosphorescence light.

It is worth noting that the results obtained by the authors to whom reference has been made above, as well as those of the more recent measurements by Lewis, Feofilov and Orndorff [7], and Mitra [5], coincide in this that, for exciting wave lengths $\lambda > 4000$ Å, the degree of polarization of fluorescence of fluorescein, rhodamine B, eosin, and similar dyes remains constant.

The aim of the present paper is to investigate the degree of polarization of fluorescence in a case in which phosphorescence is present. Instead of the degree of polarization p emission anisotropy r, as introduced by A. Jabłoński [8], will be used. For polarized exciting light

$$r = \frac{I_{\parallel} - I_{\perp}}{I_{\parallel} + 2I_{\perp}}, \tag{1}$$

and

$$r = \frac{2p}{3-p}. \tag{2}$$

The graph in Fig. 1 shows the results of measurements of the emission anisotropy of a mixture of fluorescence and phosphorescence of fluorescein in boric acid, *versus* the exciting wave length. The emission anisotropy was measured directly with an Arago compensator and Savart plate as shown schematically in [9]; however, instead of a mercury lamp, a 6 V — 30 W bulb was used, the light of which was rendered monochromatic by means of a Zeiss mirror monochromator. A spectral breadth of 50 Å was used for excitation. The Kawski-Neumann formula [10] was transformed as follows to yield the emission anisotropy as a function of the angle α of the total depolarization in the Arago compensator:

$$(3)\qquad r = \frac{2\left(1-\frac{1}{n^2}\right)\sin^2\alpha}{3\left(1+\frac{1}{n^2}\right)\left[2-\left(1+\frac{1}{n^2}\right)\sin^2\alpha\right]+12\left(\frac{1-k}{k}\right)\frac{\cos\alpha}{n^2}\sqrt{n^2-\sin^2\alpha}-\left(1-\frac{1}{n^2}\right)\sin^2\alpha}.$$

The $r(\alpha)$ curves for $n = 1.516$ and $k = 2, 3, 4, 6$ (k denoting the number of plates in the Arago compensator) are shown in Fig. 2.

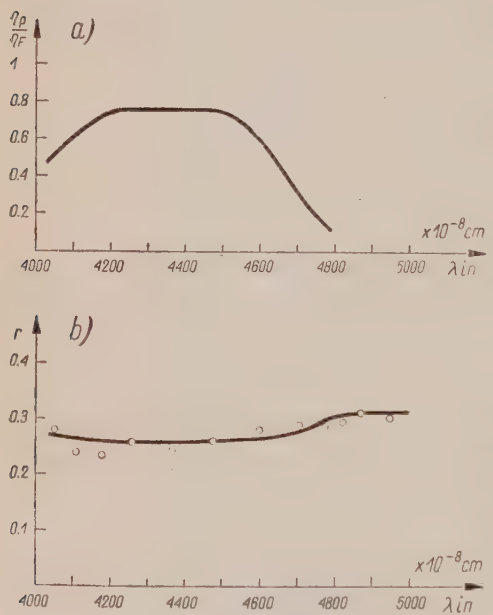

Fig. 1. a) The dependence of the emission anisotropy of total photoluminescence (fluorescence + phosphorescence) on the wave length of the exciting light
— — curve computed by means of Eq. (5)
o experimental points
b) Ratio $\frac{\eta_P}{\eta_F}$ *versus* the wave length of the exciting light (11)

On comparing the results in Fig. 1b with those of Bauer and Bączyński ([11], Fig. 1a), a rise of emission anisotropy within the region in which the phosphorescence yield falls will clearly be seen to take place. This suggested that the diminution of emission anisotropy within the region of maximum of relative phosphorescence yield might be accounted for by the presence of an admixture of weaker polarized phosphorescence light to the fluorescence light. The above conclusion was subsequently corroborated by measurements. In the case of fluorescein in boric acid, measurements of the emission anisotropy of phosphorescence light yielded $r = 0.197$ ($p = 27\,\%$). A lesser degree of polarization (and hence lower emission anisotropy) resulted from investigations by A. Jabłoński [12] on the azimuthal dependence of the degree of polarization in celophane phosphors. In these mea-

surements, the degree of polarization of the phosphorescence differed from that of the fluorescence by about 10%. Similar results were obtained by E. Laffitte [13] for rigid solutions of fluorescein, acridine yellow and trypaflavine in plexiglass; for fluorescein, $p = 28\%$ for phosphorescene and $p = 43\%$ for the fluorescene. Lewis, Magel and Lipkin [14] obtained $p = 27\%$ for phosphorescence of a rigid solution of fluorescein in sugar.

The effect of the decrease in emission anisotropy of the fluorescence accompanying the emission of the phosphorescence light may be de-

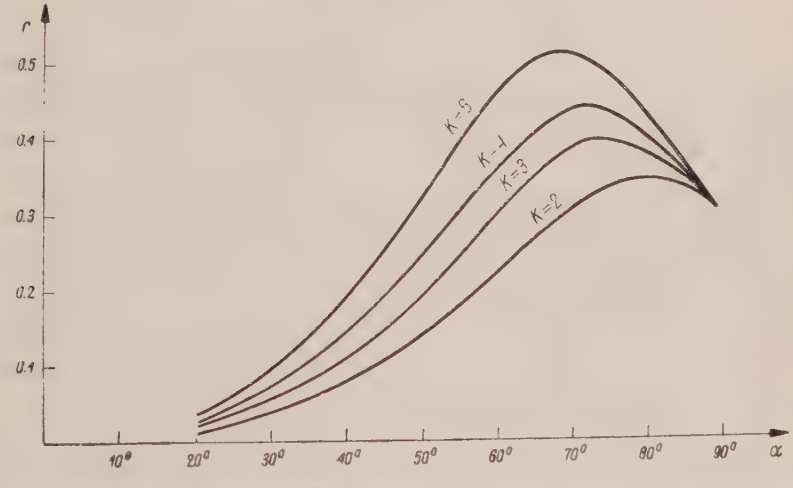

Fig. 2. The dependence of the emission anisotropy on the angle of total depolarization; k denotes the number of plates in the Arago compensator

scribed quantitatively. According to the definition of emission anisotropy:

$$r = \frac{I_{\parallel} + I'_{\parallel} - I_{\perp} - I'_{\perp}}{I_{\parallel} + 2I_{\perp} + I'_{\perp} + 2I'_{\perp}}, \tag{4}$$

where $I_{\parallel}$ and $I_{\perp}$ denote the components of the intensity of the fluorescence light parallel and perpendicular to the electric vector of the exciting light, respectively, and

$I'_{\parallel}$ and $I'_{\perp}$ — the respective components of the phosphorescence light.

Eq. (4) yields

$$r = \frac{r_F + r_P \cdot \frac{\eta_P}{\eta_F}}{1 + \frac{\eta_P}{\eta_F}}, \tag{5}$$

wherein

$$r_F = \frac{I_{\parallel} - I_{\perp}}{I_{\parallel} + 2I_{\perp}} \quad \text{and} \quad r_P = \frac{I'_{\parallel} - I'_{\perp}}{I'_{\parallel} + 2I'_{\perp}}, \tag{6}$$

r_F and r_P being the emission anisotropies of the fluorescence light and phosphorescence light respectively, and $\frac{\eta_P}{\eta_F}$ – the ratio of yields of phosphorescence and fluorescence. In [5] $\frac{I_P}{I_F}$ has been replaced by $\frac{\eta_P}{\eta_F}$ [11].

If the ratio $\frac{\eta_P}{\eta_F}$ and r_P and r are known, the emission anisotropy of pure fluorescence

$$r_F = r + \frac{\eta_P}{\eta_F}(r - r_P) \tag{7}$$

may be evaluated.

According to our experimental results r_F for fluorescein in boric acid thus evaluated remains constant for exciting wave lengths 4000 Å $< \lambda <$ 5200 Å.

Several writers obtained similar results without introducing any correction for admixture of phosphorescence. Probably the proportion of phosphorescence intensity, if any, was too small in their experiments to produce a perceptible effect.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

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Л. ЛУКАШЕВИЧ, АККУМУЛЯЦИЯ ПОГРЕШНОСТЕЙ В ПРИБЛИЖЕННЫХ ВЫЧИСЛЕНИЯХ . . . . . стр. 233–239

Приводятся формулы для исчисления суммарных результатов погрешностей, возникших при выполнении некоторого количества операций с ограниченной точностью. Дается некоторая интерпретация распределения вероятностей погрешности в аналоговых и цифровых расчетах, а также даются предположения, при которых это распределение имеет нормальный вид. Приводятся примеры для применения.

М. АЛЬТМАН, О ПРОДОЛЖЕНИИ ЛИНЕЙНЫХ ОПЕРАТОРОВ В ПРОСТРАНСТВАХ БАНАХА . . . . . стр. 241–248

В работе даны достаточные условия распространимости линейных операторов в пространствах Банаха для следующих трех случаев: 1^0 продолжаемый линейный оператор T действует из подпространства X рефлексивного пространства X_1 в X_1; 2^0 T действует из подпространства $\overline{X}_0$ пространства $\overline{X}$, сопряженного к пространству X, в $\overline{X}$; 3^0 T действует из X_0 в $\overline{X}$, где X_0 содержится в X. В качестве применения получается некоторый известный результат, касающийся распространимости билинейного функционала.

Полученные теоремы о продолжении линейных операторов применяются также к обобщенной проблеме моментов, т. е. даются достаточные условия существования линейных операторов, принимающих заданные значения в заданных точках.

Ч. БЭССАГА и А. ПЭЛЧИНСКИЙ, НЕСКОЛЬКО ЗАМЕЧАНИЙ О СОПРЯЖЕННЫХ ПРОСТРАНСТВАХ, СОДЕРЖАЩИХ ПОДПРОСТРАНСТВА ИЗОМОРФНЫЕ ПРОСТРАНСТВУ c_0 . . . . . стр. 249–250

В работе дается формулировка следующей теоремы:

Теорема. *Следующие условия равносильны:*

(i) *сопряженное пространство* X^* *содержит подпространство изоморфное* c_0,

(ii) X^* *содержит подпространство изоморфное* m,

(iii) *X содержит подпространство изоморфное пространству l, которое является проекцией X,*

(iiii) *пространство l является линейным образом X.*

Работа содержит тоже некоторые следствия из этой теоремы.

А. ПЭЛЧИНСКИЙ, СООТНОШЕНИЯ МЕЖДУ СЛАБОЙ ПОЛНОТОЙ И СЛАБОЙ БЕЗУСЛОВНОЙ СХОДИМОСТЬЮ В БАНАХОВЫХ ПРОСТРАНСТВАХ . . . . . стр. 251—253

Орлич доказал, что если пространство Банаха X—слабо полное, то всякий слабый безусловно сходящийся ряд безусловно сходится (по норме).

В первой части этой работы рассматривается класс банаховых пространств, в которых также имеет место обратная теорема.

Во второй части определяется некоторая операция „u^*-замыкания". Применяя эту операцию формулируется некоторый признак слабой полноты сопряженных пространств.

В. ЯНОВСКИЙ, О ГРАНИЧНЫХ ЗНАЧЕНИЯХ МОДУЛЯ ПРОИЗВОДНОЙ ОДНОЗНАЧНЫХ ОГРАНИЧЕННЫХ ФУНКЦИЙ . . стр. 255—259

Пусть $F_M(M>1)$ — семейство всех голоморфных и однозначных в круге $|z|<1$ функций вида

$$F(z)=z+A_2z^2+\dots, \tag{1}$$

удовлетворяющих условию $|F(z)|<M$ и F_∞—семейство всех функций вида (1).

Пусть $z\neq 0$ фиксировано. Рассматривается функционал $|F'(z)|$, определенный на F_∞ и, следовательно, для F_M.

В работе даны две оценки для $|F'(z)|$: одна для семейства F_M, вторая—для F_∞. Результат, полученный для функций семейства F_∞, совпадает с соответствующими результатами Бибербаха, доказанными другим методом.

Я. ЖЕВУСКИЙ, СПИНОРНОЕ ПРОСТРАНСТВО . . . . . стр. 261—269

В работе рассматриваются некоторые следствия предположения, что физическое пространство является скорее восьмимерным спинорным пространством, чем четырехмерным векторным пространством.

Указывается на возможность, что начало градиентного преобразования и преобразований изотопного спина проявляет геометрический характер.

А. БОНЧИНСКИЙ и М. ЧАЙКОВСКИЙ, ПОЛЯРИЗАЦИЯ ФОТОЛЮМИНЕСЦЕНЦИИ ОРГАНОФОСФОРОВ . . . . . стр. 271—274

В области возбуждений, где наряду с флуоресценцией имеет место фосфоресценция — вследствие меньшей степени поляризации этой последней — степень поляризации флуоресценции совместно с фосфоресценцией, является меньшей, чем степень поляризации самой флуоресценции. Этот эффект описан количественно применяя, вместо степени поляризации, анизотропию эмиссии.

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